数理解析研究所講究録
第 884 巻 1994 年 102-104

Modular Stacks:

Generalizing dihedral group–modular curve connections

MICHAEL D. FRIED

1991 *Mathematics Subject Classification.* Primary 11F32, 11G18, 11R58; Secondary 20B05, 20C25, 20D25, 20E18, 20F34.

Supported by NSF #DMS-99305590 and the RIMS Institute.

We thank Y. Ihara, M. Matsumoto and H. Nakamura for listening to the author justify and improve his presentation of these topics. W. Feit, R. Howe, J. G. Thompson and H. Voelklein contributed to his understanding of modular representation theory applied to the universal frattini cover. An unpublished 1987 preprint included the construction of the universal exponent p frattini covers of A_5. Others may have gone over that territory since. Serre, in particular, gave an Ext interpretation to simplify our arguments to efficient use of Lewy decompositions of projective indecomposables (see §II.B).

M. FRIED

ABSTRACT. To each finite group G we can attach a projective profinite group, $\tilde{G}$: the *universal frattini cover* of G [**FrJ**, §20.6]. Further, for any collection of r conjugacy classes $\mathbf{C}$ of G, there is a natural moduli space. Its points are equivalence classes of covers of the Riemann sphere $\mathbb{P}^1$ with geometric monodromy group G having $\mathbf{C}$ as the conjugacy classes of branch cycles of the cover. We conjoin these two constructions, applying the latter to a natural cofinal collection of finite quotients of $\tilde{G}$. This produces invariants for the arithmetic theory of curve covers. We consider here a special case that uses a prime p dividing $|G|$ and conjugacy classes of $\mathbf{C}$ of order relatively prime to p. This *p-unramified lifting invariant*, $\nu(G, p, \mathbf{C})$, is compatible with terminology of [**Se3**], to which the author contributed. We call the tower of $(G, p, \mathbf{C})$-moduli spaces that arise from this construction a *modular stack*. Arithmetic geometers know a special case: the tower of covers $X_0(p) \leftarrow X_0(p^2) \leftarrow X_0(p^3) \cdots$ of modular curves. Points on $X_0(p^n)$ correspond to pairs of elliptic curves with a cyclic p^n-power isogeny. Here G is the dihedral group D_p of order $2p$, $r = 4$ and four repetitions of the involution conjugacy class in D_p comprise $\mathbf{C}$ [**DFr**, §5.1–5.2]. The word *stack* arises from the Deligne-Mumford paper [**DeMu**]. Its subtle, yet compatible, use here has a *modular representation* interpretation.

What we understand of any modular stack comes from this lifting invariant. This lives in the p'-prime quotient of $\tilde{G}$. It is a rare, yet significant, event that the modular stack attached to $(G, p, \mathbf{C})$ may have finite length (unlike the tower of modular curves). The material from §II on the universal frattini cover (especially applied to A_n and conjugacy classes of 3-cycles) provides examples. Keeping to the p-unramified case allows these definitions an elementary elegance. To show how to generalize beyond this we compute a low level quotient of the 2-ramified invariant of L. Schnepp's 3-branch point cover of degree 20. The arithmetic applications of $\nu(G, p, \mathbf{C})$ in §IV point to a test for the *Drinfeld-Grothendieck-Ihara relations on $G_{\mathbb{Q}}$* [**I**] applied to detecting fields of definition of curve covers. A plan for that project concludes this paper.

REFERENCES

[A] J. L. Alperin, *Local representation theory*, Cambridge studies in advanced mathematics, vol. 11, Cambridge Univ. Press, 1986.

[AJL] H. Andersen, J. Jorgensen and P. Landrock, *The projective indecomposable modules* of $\mathrm{SL}_2(p^n)$, Proc. London Math. Soc. **46** (1983), 38–52.

[Be] D. Benson, *The Loewy structures for the projective indecomposable modules for A_8 and A_9 in characteristic 2*, Comm. in Alg. **11** (1983), 1395–1451.

[DFr] P. Debes and M. Fried, *Nonrigid constructions in the Inverse Galois Problem*, PJM **163 #1** (1994), 81–122.

[DeMu] P. Deligne and D. Mumford, *The irreducibily of the space of curves of given genus*, Inst. Hautes Etudes Math. **36** (1967), 75–100.

[Fr1] M. Fried, *Global construction of general xxceptional Covers: with motivation for applications to encoding*, AMS Contemporary Math series (1994).

[Fr2] M. Fried, *On a theorem of MacCluer*, Acta Arith. **25** (1974), 122–127.

[Fr3] M. Fried, *Exposition on an Arithmetic-Group Theoretic Connection via Riemann's Existence Theorem*, A.M.S. Publications, Proceedings of Symposia in Pure Math: Santa Cruz Conference on Finite Groups,, vol. 37, 1980, pp. 571–601.

[Fr4] M. Fried, *Galois groups and Complex Multiplication*, TAMS **235** (1978), 141–162.

[FJ] M.D. Fried and M. Jarden, *Field Arithmetic*, Ergebnisse der Mathematik III, vol. 11, Springer Verlag, Heidelberg, 1986.

[HaMu] J. Harris and D. Mumford, *On the Kodaira dimension of the moduli space of curves*, Invent. math. **67** (1982), 23–86.

[I] Y. Ihara, *Proceedings of the International Congress of Mathematicians*, Kyoto 1990, Springer-Verlag, Hong Kong, 1991.

[M] B. Mazur, *Lecture Notes in Mathematics*, vol. 601, Springer-Verlag, 1977.

[N] D. G. Northcott, *An introduction to homological algebra*, Cambridge Univ. Press, Great Britain, 1962.

[Se1] J.-P. Serre, *L'invariant de Witt de la forme* $T(x^2)$, Math. Helvetici **59** (1984), 651–676.

[Se2] J.-P. Serre, *Letter on Ext*, Private Correspondence (1988).

[Se3] J.-P. Serre **Comptes Rendus** (1989).

[V] P. Vojta, *Lecture Notes in Mathematics*, vol. 1239, Springer-Verlag, 1987.

UC IRVINE, IRVINE, CA 92717, USA

E-mail address: mfried@math.uci.edu